REGIONS OF STABILITY OF AN EQUATION WITH PERIODIC COEFFICIENTS

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1. Basic propositions. Let us consider the equation

$$y'' + (a - \Phi(q, \xi))\, y = 0, \qquad \Phi(q,\xi) = \Phi(q, \xi + \pi), \qquad \int_0^\pi \Phi d\xi = 0 \quad (1.1)$$

Floquet's [1-3] theory for this type equation leads to the following. Suppose that a fundamental system of solutions $y_1(\xi)$ and $y_2(\xi)$ is known, then

$$Y_1(\xi) = y_1(\xi + \pi) = a_{11} y_1(\xi) + a_{12} y_2(\xi)$$
$$Y_2(\xi) = y_2(\xi + \pi) = a_{21} y_1(\xi) + a_{22} y_2(\xi)$$

is also a fundamental system, and the C_i can be determined in such a way that $X(\xi) = C_1 Y_1(\xi) + C_2 Y_2(\xi)$ will satisfy the relation $X(\xi + \pi) = \rho X(\xi)$, where ρ is determined by the condition

$$\begin{vmatrix} a_{11} - \rho & a_{12} \\ a_{21} & a_{22} - \rho \end{vmatrix} = 0, \qquad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = 1$$

so that

$$\rho = \alpha \pm \sqrt{\alpha^2 - 1}, \qquad \alpha = \alpha(a, q) = (a_{11} + a_{22}) / 2$$

Let us now consider three possible cases.

1) The inequalities $-1 < \alpha < 1$ determine in the aq-plane a region where the solutions have the form

$$X_1(\xi) = \psi_1(\xi)\cos\frac{\theta}{2\pi} - \psi_2(\xi)\sin\frac{\theta}{2\pi}$$

$$X_2(\xi) = \psi_2(\xi)\cos\frac{\theta}{2\pi} + \psi_1(\xi)\sin\frac{\theta}{2\pi}$$

$$\psi_i(\xi) = \psi_i(\xi + \pi), \qquad \theta = \tan^{-1}\frac{(1-\alpha^2)^{1/2}}{\alpha}$$

2) In case $|\alpha| = 1$, certain lines are determined on the aq-plane. Along these lines the solutions are of period π or 2π.

$$X(\xi + \pi) = \pm X(\xi)$$

3) Finally, the inequalities $\alpha > 1$, $\alpha < -1$ determine a region in the aq-plane. In this region the solutions can be expressed in the following form:

$$X_{1,2} = \exp\left(\frac{\beta_{1,2}}{2\pi}\xi\right)L_{1,2}(\xi), \quad L_{1,2}(\xi + \pi) = L_{1,2}(\xi), \quad \beta_{1,2} = \ln|\alpha \pm (\alpha^2 - 1)^{1/2}|$$

It is not difficult to see that in the first case the solutions are non-increasing, that in the third case they are unstable in the Liapunov sense, and that the region $|\alpha| < 1$ is a region of stability of the initial equation. The curves $|\alpha| = 1$ determine the boundaries of the region of stability in the aq-plane.

Below, we give investigations of the boundaries of the regions of stability for Equation (1.1), of the periodic solutions in the region of stability.

2. Construction and properties of the boundaries of the regions of stability. Let

$$\Phi(q, \xi) = \sum_{\nu=1} \Phi_\nu(\xi) q^\nu$$

in Equation (1.1).

We shall seek the equation of the boundary of the region of stability $|\alpha| = 1$ in the form

$$a(q) = \sum_{\mu=0} a_\mu q^\mu \tag{2.1}$$

Then the solution along the resulting boundary will have the form

$$y(\xi) = \sum_{\nu=0} y_\nu(\xi) q^\nu \tag{2.2}$$

Here the $y_\nu(\xi)$ are determined by the system of equations

$$y_\mu'' + \sum_{\nu=0}^{\mu} (a_\nu - \Phi_\nu) y_{\mu-\nu} = 0, \qquad \Phi_0 = 0 \tag{2.3}$$

which is obtained as the result of substituting (2.1) and (2.2) into (1.1). The requirement $y_\nu(\xi + 2\pi) = y_\nu(\xi)$ must determine Equation (2.1) of the sought boundary, and also the periodic solution (2.2) along this boundary. The condition of periodicity will be fulfilled automatically if one seeks the solution of the system (2.3) in the form

$$y_\mu = 2 \sum_{p=0} (y_{\mu p}^{+} \cos p\xi + y_{\mu p}^{-} \sin p\xi) \tag{2.4}$$

Suppose, furthermore, that

$$\Phi_\nu = 2 \sum_{p=1} (\varphi_{\nu p}^{+} \cos 2p\xi + \varphi_{\nu p}^{-} \sin 2p\xi) \tag{2.5}$$

Then the substitution of (2.4) and (2.5) into (2.3) yields the following system of equations in $y_{\mu p}$ and a_ν:

$$\sum_{\nu=1}^{\mu} \sum_{p=1} \sum_{q=0} \{\varphi_{\nu p}^{+} y_{\mu-\nu, q}^{-} [\delta_{2p+q,m} - \delta_{2p-q,m} + \delta_{2p-q,-m}] + \tag{2.6}$$

$$+ \varphi_{\nu p}^{-} y_{\mu-\nu,q}^{+} [\delta_{2p+q,m} + \delta_{2p-q,m} - \delta_{2p-q,-m}]\} = \sum_{\nu=0}^{\mu} [a_\nu - m^2 \delta_{\nu 0}] y_{\mu-\nu,m}^{-}$$

$$\sum_{\nu=1}^{\mu} \sum_{p=1} \sum_{q=0} \{\varphi_{\nu p}^{+} y_{\mu-\nu,q}^{+} [\delta_{2p+q,m} + \delta_{2p-q,m} + \delta_{2p-q,-m}] +$$

$$+ \varphi_{\nu p}^{-} y_{\mu-\nu,q}^{-} [\delta_{2p-q,m} - \delta_{2p+q,m} + \delta_{2p-q,-m}]\} = \sum_{\nu=0}^{\mu} [a_\nu - m^2 \delta_{\nu 0}] y_{\mu-\nu,m}^{+}$$

Here δ_{mn} is Kronecker's symbol. Let us introduce the notation

$$P_{\nu q m}^{\pm} = \pm \varphi_{\nu,(m+q)/2}^{+} + \varphi_{\nu,(m-q)/2}^{+} + \varphi_{\nu,(q-m)/2}^{+} \tag{2.7}$$

$$Q_{\nu q m}^{\pm} = \varphi_{\nu,(m+q)/2}^{-} \pm \varphi_{\nu,(m-q)/2}^{-} \mp \varphi_{\nu,(q-m)/2}^{-}$$

In the new notation the system (2.6) can be rewritten in the more compact form

$$\sum_{\nu=1}^{\mu} \sum_{q=0} [Q_{\nu q m}^{\mp} y_{\mu-\nu,q}^{\mp} + P_{\nu q m}^{\pm} y_{\mu-\nu,q}^{\pm}] = \sum_{\nu=0}^{\mu} [a_\nu - m^2 \delta_{\nu,0}] y_{\mu-\nu,m}^{\pm} \tag{2.8}$$

One can make the following remarks with regard to the introduced quantities (2.7):

a) Since in (2.5) the summation starts with one, $\phi_{\nu k}^{\pm} = 0$ when $k \leq 0$; therefore, the terms $\phi_{\nu,(m-q)/2}^{\pm}$ and $\phi_{\nu,(q-m)/2}^{\pm}$ cannot occur simultaneously on the right-hand side.

b) The right-hand side vanishes if $m + q$ and, hence, also $m - q$ or $q - m$ are odd. Thus, the summation on q should actually be performed in (2.8) on even indices when m is an even number, and on odd indices if m is odd. Hence

$$P_{\nu qm}^{\pm} = Q_{\nu qm}^{\pm} = 0 \qquad (q + m = 2l + 1,\ l = 0, 1, \ldots)$$

Let us consider the solution of the system (2.8). Setting $\mu = 0$, we arrive at the following equations:

$$[a_0 - m^2]\, y_{0m}^{\pm} = 0$$

which have non-trivial solutions for $y_{0m}^{\pm}$ only if $a_0 = n^2$ (n an integer). Suppose that $a_0 = n^2 \neq 0$. Then $y_{0m}^{\pm} = 0$, $m \neq n$ and the quantities $y_{0n}^{\pm}$ are arbitrary. Setting next $\mu = 1$, we obtain the system of equations

$$P_{1nm}^{\pm} y_{0n}^{\pm} + Q_{1nm}^{\mp} y_{0n}^{\mp} = a_{1n} y_{0m}^{\pm} + (n^2 - m^2)\, y_{1m}^{\pm}$$

From this it follows that

$$y_{1m}^{\pm} = \frac{1}{n^2 - m^2} [P_{1nm}^{\pm} y_{0n}^{\pm} + Q_{1nm}^{\mp} y_{0n}^{\mp}] \quad \cdot \quad (n \neq m) \tag{2.9}$$

When $m = n$ we obtain the conditions which determine $y_{0n}^{\pm}$ and a_{1n}:

$$[P_{1nn}^{\pm} - a_{1n}]\, y_{0n}^{\pm} + Q_{1nn}^{\mp} y_{0n}^{\mp} = 0 \tag{2.10}$$

The system (2.10) admits a non-trivial solution if

$$\begin{vmatrix} Q_{1nn}^{+} & P_{1nn}^{-} - a_{1n} \\ P_{1nn}^{+} - a_{1n} & Q_{1nn}^{-} \end{vmatrix} = 0$$

Assuming that

$$\begin{vmatrix} Q_{1nn}^{+} & P_{1nn}^{-} \\ P_{1nn}^{+} & Q_{1nn}^{-} \end{vmatrix} = [\varphi_{1n}^{+}]^2 + [\varphi_{1n}^{-}]^2 \neq 0 \tag{2.11}$$

and noting that $P_{1nn}^{+} + Q_{1nn}^{-} = 0$, we find

$$a_{1n}^{j} = (-1)^j \begin{vmatrix} Q_{1nn}^{+} & P_{1nn}^{-} \\ P_{1nn}^{+} & Q_{1nn}^{-} \end{vmatrix}^{1/2} \qquad (j = 1, 2) \tag{2.12}$$

Now we can determine, up to within an arbitrary factor, the $y_{0n}^{j\pm}$ (the index j was introduced to distinguish between two possible solutions of the system (2.10)). From (2.9) we can determine (again up to within an arbitrary factor) the ${y_{1m}}^{j\pm}$. The ${y_{1n}}^{j\pm}$ remain arbitrary.

Setting $\mu = 2$, we are led to the following system:

$$\sum_q [Q^{\pm}_{1qm} y^{j\pm}_{1q} + P^{\mp}_{1qm} y^{j\mp}_{1q} + Q^{\pm}_{2qm} y^{j\pm}_{0q} + \quad (2.13)$$

$$+ P^{\mp}_{2qm} y^{j\mp}_{0q}] = a^j_{1n} y^{j\mp}_{1m} + a^j_{2n} y^{j\mp}_{0m} + (n^2 - m^2) y^{j\mp}_{2m}.$$

Whence, if $m \neq n$, we obtain

$$y^{j\pm}_{2m} = \frac{1}{n^2 - m^2} \left\{ \sum_{\nu=1}^{2} \sum_q [P^{\pm}_{\nu qm} y^{j\pm}_{2-\nu,q} + Q^{\mp}_{\nu qm} y^{j\mp}_{2-\nu,q}] - a^j_{1n} y^{j\pm}_{1m} \right\}$$

Here, the quantities on the right-hand side are known except for the ${y_{1n}}^{j\pm}$. For their determination, as well as for the determination of the ${a_{2n}}^{j}$, one can obtain the following equations by setting $m = n$:

$$\sum_{\nu=1}^{2} \sum_q [Q^{\pm}_{\nu qm} y^{j\pm}_{2-\nu,q} + P^{\mp}_{\nu qm} y^{j\mp}_{2-\nu,q}] = \sum_{\nu=1}^{2} a^j_{\nu n} y^{j\mp}_{2-\nu,n}$$

Let us rewrite the system obtained and express the unknowns explicitly:

$$\begin{aligned} Q^{+}_{1nn} y^{j+}_{1n} + [P^{-}_{1nn} - a^j_{1n}] y^{j-}_{1n} - a^j_{2n} y^{j-}_{0n} &= -[Q^{+}_{2nn} y^{j+}_{0n} + \dots] \\ [P^{+}_{1nn} - a^j_{1n}] y^{j+}_{1n} + Q^{-}_{1nn} y^{j-}_{1n} - a^j_{2n} y^{j+}_{0n} &= -[P^{+}_{2nn} y^{j+}_{0n} + \dots] \end{aligned} \quad (2.14)$$

In this form

$$\Delta(y^{j\pm}_{1n}) = \begin{vmatrix} Q^{+}_{1nn} & P^{-}_{1nn} - a^j_{1n} \\ P^{+}_{1nn} - a^j_{1n} & Q^{-}_{1nn} \end{vmatrix} = 0$$

coincides with the determinant of the system (2.10), and, therefore, in place of

$$Q^{+}_{1nn} y^{j+}_{1n} + [P^{-}_{1nn} - a^j_{1n}] y^{j-}_{1n}$$

we can introduce new variables

$$Y^j_{1n} = y^{j+}_{1n} + [Q^{+}_{1nn}]^{-1} [P^{-}_{1nn} - a^j_{1n}] y^{j-}_{1n}$$

Then the system (2.14) takes on the form

$$\begin{aligned} a^j_{2n} y^{j-}_{0n} - Y^j_{1n} Q^{+}_{1nn} &= Q^{+}_{2nn} y^{j+}_{0n} + \dots, \\ a^j_{2n} y^{j+}_{0n} - Y^j_{1n} e^j_n Q^{+}_{1nn} &= Q^{-}_{2nn} y^{j-}_{0n} + \dots \end{aligned} \quad (2.15)$$

where

$$\begin{vmatrix} Q_{1nn}^{+} & y_{0n}^{j-} \\ \varepsilon_n^j Q_{1nn}^{+} & y_{0n}^{j+} \end{vmatrix} \neq 0, \qquad \varepsilon_n^j = \frac{Q_{1nn}^{-}}{P_{1nn}^{-} - a_{1n}^j} = \frac{P_{1nn}^{+} - a_{1n}^j}{Q_{1nn}^{+}}$$

From Equations (2.15) it can be seen that ${a_{2n}}^j$ takes on an exact value (it depends on ${Q_{2nn}}^{\pm}$, ${P_{2nn}}^{\pm}$, ${y_{0n}}^{j-}/{y_{0n}}^{j+}$) while ${Y_{1n}}^j$ depends linearly on ${y_{0n}}^{j\pm}$, namely, just as ${y_{0n}}^{j\pm}$, it is determined except for a muliplicative factor. The ${y_{2n}}^{j\pm}$ remain arbitrary and will be determined at the next step.

Let us consider the case of an arbitrary μ. If $m \neq n$ we have

$$y_{\mu m}^{j\pm} = \frac{1}{n^2 - m^2}\left\{\sum_{\nu=1}^{\mu}\sum_{q}\left[P_{\nu q m}^{\pm} y_{\mu-\nu,q}^{j\pm} + Q_{\nu q m}^{\mp} y_{\mu-\nu,q}^{j\mp}\right] - \sum_{\nu=1}^{\mu-1} a_{\nu n}^j y_{\mu-\nu,m}^{j\pm}\right\} \quad (2.16)$$

On the right-hand side of (2.16) stand known quantities, except for the ${y_{\mu-1,n}}^{j\pm}$ which are determined by a system obtained from (2.8) by setting $m = n$:

$$Q_{1nn}^{+} y_{\mu-1,n}^{j+} + [P_{1nn}^{-} - a_{1n}^j]\, y_{\mu-1,n}^{j-} - a_{\mu n}^j y_{0n}^{j-} = \ldots$$
$$[P_{1nn}^{+} - a_{1n}^j]\, y_{\mu-1,n}^{j+} + Q_{1nn}^{-} y_{\mu-1,n}^{j-} - a_{\mu n}^j y_{0n}^{j+} = \ldots$$

Here, just as in the preceding equation

$$\Delta(y_{\mu-1,n}^{j\pm}) = \begin{vmatrix} Q_{1nn}^{+} & P_{1nn}^{-} - a_{1n}^j \\ P_{1nn}^{+} - a_{1n}^j & Q_{1nn}^{-} \end{vmatrix} = 0$$

Therefore, setting

$$Y_{\mu-1,n}^j = y_{\mu-1,n}^{j+} + [Q_{1nn}^{+}]^{-1}[P_{1nn}^{-} - a_{1n}^j]\, y_{\mu-1,n}^{j-}$$

we obtain a system in terms of ${a_{\mu n}}^j$ and ${Y_{\mu-1,n}}^j$

$$y_{0n}^{j-} a_{\mu n}^j - Q_{1nn}^{+} Y_{\mu-1,n}^j = \ldots \qquad y_{0n}^{j+} a_{\mu n}^j - Q_{1nn}^{+} \varepsilon_n^j Y_{\mu-1,n}^j = \ldots \quad (2.17)$$

where on the right-hand side are terms which are linear in the ${y_{\nu k}}^{j\pm}$. From the system (2.17) it can be seen that the ${a_{\mu n}}^j$ have exact values; the quantities ${Y_{\mu-1,n}}^j$ can be determined to within the above-mentioned factor, and hence ${Y_{\mu-1,n}}^{j\pm}$ are also determinable to within this factor, while the ${y_{\mu n}}^{j\pm}$ still remain arbitrary.

Thus, if $[{\phi_{1n}}^{+}]^2 + [{\phi_{1n}}^{-}]^2 \neq 0$, the method developed above makes it possible to construct the equations of the boundaries of the regions of stability in the form

$$a_n^j = n^2 + \sum_{\mu=1} a_{\mu n}^j q^\mu \tag{2.18}$$

and to determine the periodic solutions along these boundaries

$$y^j = 2\sum_{p\mu} (y_{\mu p}^{j+} \cos p\xi + y_{\mu p}^{j-} \sin p\xi)\, q^\mu \tag{2.19}$$

The formulas obtained permit one to make several general assertions relative to the properties of the periodic solutions and the boundaries of the regions of stability of the initial equation. Starting from the fact that $P_{\nu q m}{}^{\pm}$ and $Q_{\nu q m}{}^{\pm}$ are different from zero when q and m are both even or both odd, we conclude that the solutions which can exist along the curves $a = a_n{}^j(q)$, and which for $q = 0$ pass through $a(0) = n^2$ (n even), contain only even harmonics $y_{\nu k}{}^{\pm}$, that is, they are periodic of period π. Indeed, from (2.9) it follows that the $y_{1m}{}^{j\pm}$ are different from zero only for even m. The formulas for $y_{2m}{}^{j\pm}$ also show that m is even, for the summation is carried out in them only over even q. By assuming that up to $\mu - 1$ the solutions contain only even harmonics, we can conclude on the basis of (2.16) that the μth solution has the same property, which proves the truth of the stated assertion. An analogous assertion is true for odd n. From the results obtained it follows also that the region which lies between the curves $a_m{}^{1,2}$ and $a_{n+1}{}^{2,1}$, and contains the axis $q = 0$, is a region of stability, since along the axis $q = 0$ the original equation has a stable solution.

We note that through the point (0.0) in the aq-plane there passes a curve $a = a_0(q)$ below which there are no regions of stability. Indeed, when $\mu = 0$, we have the solution $y_{0m}{}^{\pm} = 0$, $m \neq 0$; the quantity $y_{00}{}^{+}$ is arbitrary. The quantity $y_{00}{}^{-} = 0$ because the original equation does not have an odd periodic solution. The following system ($\mu = 1$) has a solution of the form

$$y_{1m}^{+} = -\frac{1}{m^2} P_{10m}^{+} y_{00}^{+}, \qquad y_{1m}^{-} = -\frac{1}{m^2} Q_{10m}^{+} y_{00}^{+}$$

Setting, finally, $\mu = 2$, we are led to a system which determines the $y_{2,m}{}^{\pm}$ ($m \neq 0$). When $m = 0$, this system takes the form

$$-a_{20} y_{00}^{+} = -\sum_q P_{1q0}^{+} y_{1q}^{+} + Q_{1q0}^{-} y_{1q}^{-}, \qquad 0 = \sum_q Q_{1q0}^{+} y_{1q}^{+} + P_{1q0}^{-} y_{1q}^{-}$$

Since $Q_{\nu q0}{}^{+} = P_{\nu q0}{}^{-} = 0$, the second sum vanishes identically. For a_{20} we obtain

$$a_{20} = -\sum_q \frac{1}{q^2} \{[P_{10q}^+]^2 + [Q_{10q}^+]^2\}$$

because $P_{\nu q0}^+ = P_{\nu 0q}^+$ and $Q_{\nu q0}^- = Q_{\nu 0q}^+$. From here on the solution is carried out the same way as in the case $n \neq 0$.

Before we pass to the consideration of the case when the condition (2.11) is violated, let us note that the results can be considerably simplified if $\Phi(q, \xi) = \Phi(q, -\xi)$. In this case $Q_{\nu q n_+}^{\pm} \equiv 0$, and the system (2.8) can be split into two systems relative to $y_{\mu q}^+$ and $y_{\mu q}^-$, taking on the form

$$\sum_{\nu=1}^{\mu} \sum_q P_{\nu q m}^{\pm} y_{\mu-\nu, q}^{\pm} = \sum_{\nu=1}^{\mu} a_{\nu n}^{1,2} y_{\mu-\nu, m}^{\pm} + (n^2 - m^2) y_{\mu m}^{\pm} \tag{2.20}$$

If one introduces into the discussion the quantities

$$\lambda_{\mu q}^{\pm} = [y_{0n}^{\pm}]^{-1} y_{\mu q}^{\pm}, \qquad \psi_{\mu m}^{\pm} = \sum_{\nu=1}^{\mu} \sum_q P_{\nu q m}^{\pm} \lambda_{\mu-\nu, q}^{\pm}$$

then the general solution can be expressed in the form

$$\lambda_{\mu m}^{\pm} = \frac{1}{n^2 - m^2} \left\{\psi_{\mu m}^{\pm} - \sum_{\nu=1}^{\mu} \psi_{\nu n}^{\pm} \lambda_{\mu-\nu, m}^{\pm}\right\}, \qquad \psi_{\mu m}^{\pm} = \sum_{\nu=1}^{\mu} \sum_q P_{\nu q m}^{\pm} \lambda_{\mu-\nu, q}^{\pm} \tag{2.21}$$

The results $a_n^{1,2}$ and $y_n^{\pm}$ then become

$$a_n^{1,2} = \sum_{\mu=1} \psi_{\mu n}^{\pm} q^{\mu} + n^2, \qquad y_n^{\pm} = y_{0n}^{\pm} \left\{\sum_{\mu=0} \sum_{\nu=0} \lambda_{\mu\nu}^{\pm} \begin{Bmatrix} \cos \nu\xi \\ \sin \nu\xi \end{Bmatrix} q^{\mu}\right\} \tag{2.22}$$

Let us now consider the case when condition (2.11) is satisfied. In accordance with (2.12), $a_{1n}^j = 0$, and therefore all the terms which contain $y_{1n}^{j\pm}$ will vanish in the system (2.14). We must require now that this system has a non-trivial solution y_{0n}. For this purpose we express the terms remaining on the right-hand side in terms of $y_{0n}^{j\pm}$ (in accordance with (2.9)). Thus, we obtain

$$L_n y_{0n}^{j+} + [M_n - a_{2n}^j] y_{0n}^{j-} = 0, \quad [N_n - a_{2n}^j] y_{0n}^{j+} + R_n y_{0n}^{j-} = 0$$

Here

$$L_n = \sum_q \frac{1}{n^2 - q^2} \{P_{1qn}^- Q_{1nq}^+ + P_{1nq}^+ Q_{1qn}^+\} + Q_{2nn}^+$$

$$M_n = \sum_q \frac{1}{n^2 - q^2} \{P^-_{1qn} P^-_{1nq} + Q^+_{1qn} Q^-_{1nq}\} + P^-_{2nn}$$

$$N_n = \sum_q \frac{1}{n^2 - q^2} \{P^+_{1qn} P^+_{1nq} + Q^-_{1qn} Q^+_{1nq}\} + P^+_{2nn}$$

$$R_n = \sum_q \frac{1}{n^2 - q^2} \{P^+_{1qn} Q^-_{1nq} + Q^-_{1qn} P^-_{1nq}\} + Q^-_{2nn}$$

Hence, the condition

$$\begin{vmatrix} L_n & M_n - a^j_{2n} \\ N_n - a^j_{2n} & R_n \end{vmatrix} = 0$$

will yield two values for ${a_{2n}}^j$. After this we find ${y_{0n}}^{j\pm}$ except for an arbitrary factor. In conclusion, we note that the violation of the condition (2.11) indicates the absence of the $2n$th harmonic in the function $\Phi_1(\xi)$, and leads to an expansion of the region of stability near the point $q = 0$, $a(0) = n^2$ (in Fig. 1, condition (2.11) holds; in Fig. 2, the condition (2.11) is not satisfied).

3. Construction of solutions within the region of stability. Within the considered regions of stability the form of the solutions given by Floquet's theory can be made more precise. The method developed above permits one to construct solutions of period $2\pi s$, which are realized in the region of stability along the curves $a = a(q)$ which pass through the points $a|_{q=0} = (l/s)^2$, where (l/s) is an irreducible fraction.

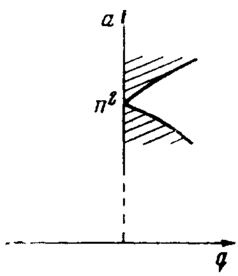

Fig. 1.

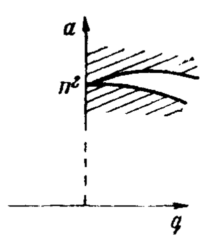

Fig. 2.

As above, we shall look for a solution, along the indicated curves, in the form

$$y^s_\mu = \sum_q \left(y^{s+}_{\mu q} \cos \frac{q}{s} \xi + y^{s-}_{\mu q} \sin \frac{q}{s} \xi \right) \tag{3.1}$$

The equations of the curves along which such solutions can exist we write in the form

$$a^s(q) = \sum_{\nu=1} a_\nu{}^s q^\nu + \left(\frac{l}{s}\right)^2 \tag{3.2}$$

Then the discussions which led to the construction of the stability regions lead to the following system of equations:

$$\sum_{\nu=1}^{\mu} \sum_q [P_{\nu qm}^{s\pm} y_{\mu-\nu,q}^{s\pm} + Q_{\nu qm}^{s\mp} y_{\mu-\nu,q}^{s\mp}] = \sum_{\nu=1}^{\mu} a_\nu^s y_{\mu-\nu,m}^{s\pm} + \left[\left(\frac{l}{s}\right)^2 - \left(\frac{m}{s}\right)^2\right] y_{\mu m}^{s\pm} \tag{3.3}$$

Here, in analogy with (2.7), we have introduced the notation

$$\begin{aligned} P_{\nu qm}^{s\pm} &= \pm\varphi_{\nu,(q+m)/2s}^{+} + \varphi_{\nu,(m-q)/2s}^{+} + \varphi_{\nu,(q-m)/2s}^{+} \\ Q_{\nu qm}^{s\pm} &= \varphi_{\nu,(q+m)/2s}^{-} \pm \varphi_{\nu,(m-q)/2s}^{-} \mp \varphi_{\nu,(q-m)/2s}^{-} \end{aligned} \tag{3.4}$$

In spite of the external similarity of the system obtained with the system which determines the equations of the boundaries of the stability regions, the nature of the solutions of the obtained system is quite different. In order to establish this, let us consider the properties of the number $P_{\nu qm}{}^{s\pm}$ and $Q_{\nu qm}{}^{s\pm}$ (which for $s = 1$ coincide with the terms $P_{\nu qm}{}^{\pm}$ and $Q_{\nu qm}{}^{\pm}$ introduced earlier. On account of (3.4) one can easily see that

$$P_{\nu qm}^{s\pm} = P_{\nu mq}^{s\pm}, \qquad Q_{\nu qm}^{s+} = Q_{\nu mq}^{s-}$$

This system of relations is valid for arbitrary s (in particular, for $s = 1$). However, if $s \neq 1$, one obtains a set of relations which are not true for the case when $s = 1$. Let us consider the terms $P_{\nu qm}{}^{s\pm}$ and $Q_{\nu qm}{}^{s\pm}$ in which one of the indices q or m is equal to l, when, as above, l/s is an irreducible fraction, i.e. $l \neq ks$, $k = 0, 1, \ldots$. Then one can show that on the right-hand side of (3.4) only one of the terms can be distinct from zero. Indeed, the index t of any term $\phi_{\nu t}{}^{\pm}$ distinct from zero must be a positive integer. Hence, the only terms which can be simultaneously different from zero are either $\phi_{\nu,(m+q)/2s}^{\pm}$ and $\phi_{\nu,(m-q)/2s}^{\pm}$ or $\phi_{\nu,(m+q)/2s}^{\pm}$ and $\phi_{\nu,(q-m)/2s}^{\pm}$. Of these terms only one is found to be distinct from zero if $m = l$ or $q = l$. Indeed, suppose $\phi_{\nu,(l+q)/2s}^{\pm}$ and $\phi_{\nu,(l-q)/2s}^{\pm}$ are different from zero. Then it is necessary that $l + q = 2k_1 s$ and $l - q = 2k_2 s$, where k_1 and k_2 are integers. From this it follows that $l = (k_1 + k_2)s$, which contradicts the previously agreed condition that $l \neq ks$. If one assumes that the second pair is distinct from zero, then the requirement that $l + q = 2k_1 s$ and $q - l = 2k_2 s$ leads again to the contradiction that $l = (k_1 - k_2)s$. These conclusions make it possible to note a number of special properties of the numbers (3.4). We find that

$$[P^{s+}_{\nu ql}]^2 = [P^{s-}_{\nu ql}]^2 = [P^{s+}_{\nu lq}]^2 = [P^{s-}_{\nu lq}]^2 \tag{3.5}$$

and also

$$[Q^{s+}_{\nu lq}]^2 = [Q^{s+}_{\nu ql}]^2 = [Q^{s-}_{\nu lq}]^2 = [Q^{s-}_{\nu ql}]^2 \tag{3.6}$$

Next, suppose that $\phi^{\pm}_{\nu,(l+q)/2s}$ is distinct from zero. Then

$$P^{s+}_{\nu ql} = -P^{s-}_{\nu ql}, \qquad Q^{s+}_{\nu ql} = Q^{s-}_{\nu ql} \tag{3.7}$$

If, however, $\phi^{\pm}_{\nu,\pm(l-q)/2s}$ is different from zero, then we have

$$P^{s+}_{\nu ql} = P^{s-}_{\nu ql}, \qquad Q^{s+}_{\nu ql} = -Q^{s-}_{\nu ql} \tag{3.8}$$

Equations (3.7) and (3.8) cannot hold simultaneously.

Let us consider the solutions of (3.3). For $\mu = 1$ we obtain

$$P^{s\pm}_{1lm}\, y^{s\pm}_{0l} + Q^{s\mp}_{1lm} y^{s\mp} = a_1{}^s y^{s\pm}_{0l}\delta_{ml} + \frac{l^2-m^2}{s^2}\, y^{s\pm}_{1m} \tag{3.9}$$

From this, with $m \neq l$, it follows that

$$y^{s\pm}_{1m} = s^2\, \frac{P^{s\pm}_{1lm} y^{s\pm}_{0l} + Q^{s\mp}_{1lm} y^{s\mp}_{0l}}{l^2-m^2} \tag{3.10}$$

In case, however, that $m = l$ we find that $a_1{}^s = 0$, since $P_{1ll}{}^{s\pm} = Q_{1ll}{}^{s\pm} = 0$.

Setting $\mu = 2$, we obtain a system of equations of the form

$$\sum_q [P^{s\pm}_{1qm} y^{s\pm}_{1q} + Q^{s\mp}_{1qm} y^{s\mp}_{1q}] + P^{s\pm}_{2lm} y^{s\pm}_{0l} + Q^{s\mp}_{2lm} y^{s\mp}_{0l} = a^s_2 y^{s\pm}_{0l}\delta_{lm} + \frac{l^2-m^2}{s^2} y^{s\pm}_{2m} \tag{3.11}$$

Substituting Expression (3.10) into the system (3.11), we are led to a system of equations in $y_{0l}{}^s$ (homogeneous for $m = l$) from which we require that it have a non-trivial solution when $m = l$. The last requirement must determine $a_2{}^s$. If we make the indicated substitution, we arrive at the following relation:

$$\sum_q{}' \frac{[P^{s\pm}_{1ql} P^{s\pm}_{1lq} + Q^{s\mp}_{1ql} Q^{s\pm}_{1lq}]\, y^{s\pm}_{0l} + [P^{s\pm}_{1ql} Q^{s\mp}_{1lq} + Q^{s\mp}_{1ql} P^{s\mp}_{1lq}]\, y^{s\mp}_{0l}}{(l^2-q^2)/s^2} = a^s_2 y^{s\pm}_{0l} \tag{3.12}$$

The above-mentioned properties of the numbers P and Q make it possible to establish that

$$P^{s+}_{1ql}P^{s+}_{1lq} \equiv P^{s-}_{1lq}P^{s-}_{1ql} \equiv [P^{s\pm}_{1lq}]^2, \qquad Q^{s-}_{1lq}Q^{s+}_{1ql} \equiv Q^{s-}_{1ql}Q^{s+}_{1lq} \equiv [Q^{s\pm}_{1lq}]^2$$

and also that

$$P^{s+}_{1ql}\ Q^{s-}_{1lq} + Q^{s-}_{1ql}P^{s-}_{1lq} = P^{s-}_{1ql}\ Q^{s+}_{1lq} + P^{s+}_{1lq}Q^{s+}_{1ql} = 0$$

These identities which are false in case $s = 1$ alter the solutions of the system, which now has only one solution if

$$a_2^s = \sum_q \frac{[P^{s\pm}_{1lq}]^2 + [Q^{s\pm}_{1lq}]^2}{(l^2 - q^2)/s^2}$$

And thus, the $y_{0l}{}^{s\pm}$ remain arbitrary and are not interconnected. The remaining computations do not present any difficulties (they proceed as in the case of the finding of the boundaries), and lead to the construction of two independent solutions (this is different from the case of finding the boundaries, where along each boundary there was found only one periodic solution).

Thus, the curve, along which there can exist a solution of period $2\pi s$, is given up to within the second approximation by the equation

$$a^s = \left(\frac{l}{s}\right)^2 + \left\{\sum_p \frac{[P^{s\pm}_{1lp}]^2 + [Q^{s\pm}_{1lp}]^2}{(l^2 - p^2)/s^2}\right\} q^2 + \dots \qquad (3.13)$$

On the other hand, if condition (2.11) is satisfied, the equation of the boundary passing through the point $a|_{q=0} = n^2$ can be written in the form

$$a_n^j = n^2 + (-1)^j \begin{vmatrix} Q^+_{1nn} & P^-_{1nn} \\ P^+_{1nn} & Q^-_{1nn} \end{vmatrix}^{1/2} q + \dots \qquad (3.14)$$

Since the last expansion begins with linear terms in q, there arises the question whether the curves (3.13) and (3.14) can intersect if l/s is near n, while q is sufficiently small. We shall show that this cannot happen.

The condition of sufficient nearness of l/s to n we shall write in the form

$$l/s = n + \theta/s \qquad (3.15)$$

where θ is a positive or negative integer. If one selects s large enough, such that $|\theta/s| < \epsilon$ (θ is fixed), then $|l/s - n| < \epsilon$. (In Equation (3.15) $\theta < 0$ when $j = 1$, and $\theta > 0$ when $j = 2$). For the proof we note that $P_{\nu lp}^{s\pm} = P_{\nu nn}^{\pm}$ and $Q_{\nu lp}^{s\pm} = Q_{\nu nn}^{\pm}$ when $l = ns + \theta$ and $p = ns - \theta$. After this it is not difficult to see that the largest term in

$$\sum_p \frac{[P_{1lp}^{s\pm}]^2 + [Q_{1lp}^{s\pm}]^2}{(l^2 - p^2)/s^2}$$

will, for a sufficiently large s, be

$$\{[P_{1nn}^{+}]^2 + [Q_{1nn}^{+}]^2\} \frac{s}{4n\theta}$$

Therefore, one can investigate the intersection of (3.14) with

$$a^{*s} = \left(\frac{l}{s}\right)^2 + \{[P_{1nn}^{+}]^2 + [Q_{1nn}^{+}]^2\} \frac{s}{4n\theta} q^2$$

But these curves cannot intersect since the equation $a_n^{\ j} = a^{*s}$ does not have a real root, as is easily verified. This completes the proof.

Thus, the structure of the stability regions is as follows: the region of stability is everywhere densely filled with curves of the form (3.13); along these curves there can exist two linearly independent solutions of period $2\pi s$. These curves intersect the axis $q = 0$ orthogonally at the points $a|_{q=0} = (l/s)^2$, where l/s is an irreducible fraction.

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BIBLIOGRAPHY

1. McLachlan, N.W., *Theory and Applications of Mathieu Functions.* (Russian translation, IIL, 1953). Oxford, 1947.

2. Brillouin, L. and Parodi, M., *Propagation des ondes dans les milieux périodiques.* (Russian translation, IIL, 1959). Paris, Mason, 1956.

3. Whittaker, E.T. and Watson, D.N., *A Course of Modern Analysis.* (Russian translation, Part 2, GTTI, 1934). Cambridge.

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